European actuarial academy

an initiative of the 'Deutsche Aktuarvereinigung', the Dutch 'Actuarieel Genootschap', the 'Schweizerische Aktuarvereinigung' and the 'Aktuarvereinigung Österreichs'

Session 5

Introduction to stochastic IBNR-methods

Agenda

- Stochastic models versus mechanical methods
- Variability of predictions
- Stochastic models underlying mechanical methods
- Poisson model

S tochastic models versus mechanical methods

Mechanical methods/techniques ('deterministic')

- Algorithmic calculation of reserves
- Only best estimate of ultimate loss, no measure of precision
- Informal assessment of fit to data

'Stochastic'

- Specification of probability distribution or mean+(co)variance
- Testing fit to data:
  which (sub)model describes reality best?
- Prediction error available
- Diagnostics: which developments deviate significantly?
- Both quantitative and qualitative assessment
  (added value of actuary)

Why stochastic IBNR-models?

Stochastic IBNR-models have become increasingly popular:

- best estimates not sufficient anymore
  => more focus on accuracy of predictions
- incorporated in user-friendly actuarial software programs
  (e.g. model selection, data handling)
- progress in literature

first step: assumptions about data $\Rightarrow$ underlying model

Types of stochastic IBNR-models

- Linear regression
  - normal
  - log-normal (Zehnwirth)
  - autoregressive (Mack)
- Generalized linear models (Renshaw & Verrall)
- Structure models (Posthuma partners)
- Bayesian models
  - Bornhuetter-Ferguson
- Bootstrapping (e.g. to mechanical methods)

Accuracy

Variability of predictions

1. Prediction error
   - Standard deviation of possible reserve outcomes

2. Predictive distribution
   - Entire distribution of possible outcomes

Mean-variance relations

Normal	$E[D_{ij}] = \mu_{ij}$ and $Var[D_{ij}] = \sigma_{ij}^2$
Poisson	$E[D_{ij}] = \mu_{ij}$ and $Var[D_{ij}] = \varphi\mu_{ij}$
Gamma	$E[D_{ij}] = \mu_{ij}$ and $Var[D_{ij}] = \varphi\mu_{ij}^2$
Inverse Gaussian	$E[D_{ij}] = \mu_{ij}$ and $Var[D_{ij}] = \varphi\mu_{ij}^3$

Decomposition prediction variance

Prediction variance

=

process variance + estimation variance

Mean Squared Prediction Error

$$E[(Y-\hat{Y})^2] = E[(\{Y-E[Y]\} - \{\hat{Y}-E[Y]\})^2]$$

$$=^{*} E[(Y-E[Y])^2] + E[(\hat{Y}-E[\hat{Y}])^2] - 2\,E[(Y-E[Y])(\hat{Y}-E[\hat{Y}])]$$

$$=^{**} E[(Y-E[Y])^2] + E[(\hat{Y}-E[\hat{Y}])^2]$$

* assuming $\hat{Y}$ is unbiased

** assuming future and past observations are independent

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Stochastic models underlying mechanical methods

Some stochastic models have identical best estimate reserve as mechanical methods

multiplicative Poisson – loss (↓) and development (→) period
⇒ chain-ladder method

autoregressive model – Mack
⇒ chain-ladder method

multiplicative Poisson – loss (↓) and calendar (↘) period
⇒ separation method (arithmetical)

log-normal – loss (↓) and calendar (↘) period
⇒ separation method (geometrical)

Different stochastic models can encompass same method

Stochastic Chain Ladder Models

Mack

- autoregressive $E[X_{ij} \mid X_{i,j-1}] = f_{j-1} X_{i,j-1}$
- distribution-free (first two moments)

Renshaw en Verrall

- Poisson distribution: $X_{ij} \sim \mathrm{Po}(\alpha_i \times \beta_j)$
- X_{ij} are stochastically independent
- entire distribution

⇒ two models corresponding to one method

Poisson model (1)

Define D_{ij} as incremental claim in loss period i and development period j

Poisson model: $D_{ij} \sim \text{Poisson}(\lambda_{ij})$

$P[D_{ij}=d] = \exp(-\lambda_{ij}) \times \lambda_{ij}^{d} / d!$

$E[D_{ij}] = \lambda_{ij} = \alpha_i \times \beta_j$ $\quad$ $Var[D_{ij}] = \lambda_{ij} = \alpha_i \times \beta_j$

"Overdispersed" Poisson: $Var[D_{ij}] = \kappa \times \alpha_i \times \beta_j$

$\Rightarrow$ Quasi-likelihood: not only for claim numbers (integers)

Poisson model (2)

α_i : expected ultimate loss period i

β_j : proportion of ultimate loss in development period j

β_j must be positive $\Rightarrow$ positive column totals (see later on)

Restriction on parameters, e.g. $\Sigma\, \beta_j = 1$ (or $\alpha_1=\beta_1=0$)

Other possibility:

$\lambda_{ij} = \alpha_i \times \beta_j \times \gamma_{i+j}$ ($\Rightarrow$ including calendar-effect)

Poisson model (3) – maximum likelihood estimator

Principle of maximum likelihood estimation

- Take joint probability-function of data: $L = \prod_{i,j} P[D_{ij}=d_{ij}]$ ('likelihood-function')
- Insert realised values of d_{ij}
- Maximize function with respect to unknown parameters (find distribution with highest probability on realised data)

Analytically and numerically easier: maximize log-likelihood

$$\ell = \ln(L) = \sum_{i,j} \ln(P[D_{ij}=d_{ij}])$$

Poisson model (4) – maximum likelihood estimator

Log-likelihood function:

$$\ell = \sum_{i=1}^{n} \sum_{j=1}^{\overline{m}_i} -\lambda_{ij} + d_{ij} \ln(\lambda_{ij}) \qquad \lambda_{ij} = \alpha_i \times \beta_j$$

Maximization by putting derivatives equal to zero:

$$\frac{\partial \ell}{\partial \alpha_k} = 0 \Rightarrow -\sum_{j=1}^{m_k} \beta_j + \sum_{j=1}^{m_k} \frac{d_{kj}}{\alpha_k \beta_j} \beta_j = 0$$

$$\Leftrightarrow \hat{\alpha}_k = \sum_{j=1}^{m_k} d_{kj} \Big/ \sum_{j=1}^{m_k} \hat{\beta}_j$$

Poisson model (5) – maximum likelihood estimator

Similarly: $$\hat{\beta}_l = \sum_{i=1}^{m_l^*} d_{il} \Big/ \sum_{i=1}^{m_l^*} \hat{\alpha}_i$$

Basic algorithm to solve:

1) take starting values for β's (e.g. 1,0,...,0)
2) calculate α's
3) calculate β's
4) repeat 2) and 3) until convergence

However: quicker and more stable numerical algorithms exist!

Poisson model (6) – maximum likelihood estimator

Property of Poisson ML-estimators

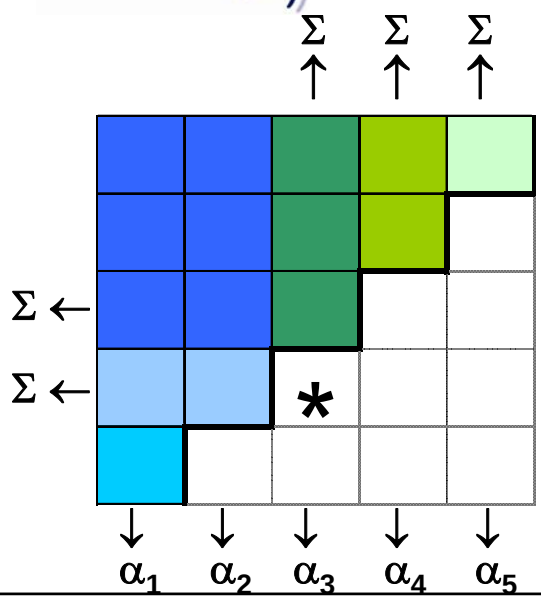
(back) estimated row/column totals

=

triangle row/column totals

$$\hat{\alpha}_k = \sum_{j=1}^{m_k} d_{kj} \Big/ \sum_{j=1}^{m_k} \hat{\beta}_j \quad \Leftrightarrow \quad \boxed{\sum_{j=1}^{m_k} \underbrace{\hat{\alpha}_k \hat{\beta}_j}_{\hat{\lambda}_{kj}} = \sum_{j=1}^{m_k} d_{kj}}$$

Poisson model (7) – resemblance with CL



$$D_{ij} \sim \mathrm{Po}(\alpha_i \times \beta_j)$$

$$\hat{\alpha}_3 = \frac{\square}{\text{const}}$$

$$\hat{\beta}_4 = \frac{\square}{\text{const}}$$

$$\hat{\alpha}_3\hat{\beta}_4 = \frac{\square \times \square}{\text{const}}$$

Poisson model (8) – smoothing

- Many parameters in the model: for each row/column
  $\Rightarrow$ estimation variance
- Losses have not been fully settled during data period
  $\Rightarrow$ prediction beyond square array
- Parsimony of parameters:
  - testing on significance and equality of parameters (likelihood ratio test)
  - smoothing development pattern with curve, e.g.

$$\ln(\lambda_{ij}) = \mu + \alpha_i + \beta_i \ln(j) + \gamma_i j$$

Poisson model (9) – prediction error

$$\mathrm{MSEP}_{ij} = \mathrm{Var}[D_{ij}] + \mathrm{Var}[\hat{D}_{ij}]$$

$$\approx \hat{\kappa}\,\hat{\lambda}_{ij} + \hat{\lambda}_{ij}^2\,\mathrm{Var}[\hat{\mu} + \hat{\alpha}_i^* + \hat{\beta}_j^*]$$

$$\mathrm{MSEP}_{total} \approx \sum \mathrm{MSEP}_{ij} + 2 \sum \lambda_{ij}\,\lambda_{kl}\,\mathrm{Cov}[\hat{\mu}+\hat{\alpha}_i^*+\hat{\beta}_j^*,\ \hat{\mu}+\hat{\alpha}_k^*+\hat{\beta}_l^*]$$

$\Rightarrow$ most software packages calculate quantities needed (e.g. S-PLUS, GLIM)