European actuarial academy

an initiative of the 'Deutsche Aktuarvereinigung', the Dutch 'Actuarieel Genootschap', the 'Schweizerische Aktuarvereinigung' and the 'Aktuarvereinigung Österreichs'

Session 6

Mack's model

Bootstrapping

Agenda

- Introduction to the model of Mack
- Predictive distribution: Bootstrapping

Assumptions Mack's model (1)

Define C_{ij} as <u>cumulative</u> claim in loss period i and development period j

1. $E[C_{i,j+1} \mid C_{i1},\ldots,C_{ij}] = f_j C_{ij}$

2. Loss periods i are stochastically independent

3. $Var[C_{i,j+1} \mid C_{i1},\ldots,C_{ij}] = \sigma_j^2 C_{ij}$

See: T. Mack (1993) "Distribution-free calculation of the standard-error of chain-ladder reserve estimates", ASTIN Bulletin

Assumptions Mack's model (2)

- Tail-factor can be included, but estimates needed of:
  - factor itself
  - variance of estimate

  ⇒ professional judgment of actuary
  ⇒ similar to Bornhuetter-Ferguson approach

- Comparison to Poisson model:

  ⇒ Mack's model does not have symmetry-property

Results of Mack (1)

Chain-ladder factors

$$\hat{f}_k = \sum_i C_{i,k+1} \Big/ \sum_i C_{ik}$$

are

- unbiased
- uncorrelated
- minimum variance

estimators of f_k

Results of Mack (2)

Prediction error of loss period *i* is approximately:

$$\text{MSEP}_i \approx \hat{C}_{i,ult}^2 \sum_j \frac{\hat{\sigma}_j^2}{\hat{f}_j^2} \left(\frac{1}{\hat{C}_{ij}} + \frac{1}{\sum_k C_{kj}} \right)$$

with

$\hat{C}_{i,ult}$: estimator of ultimate loss of accident period *i*

$\hat{\sigma}_j^2$: estimator of variance (in assumption 3)

$\Rightarrow$ Mack (1999) proposes recursion formula

Results of Mack (3)

$$\mathrm{Var}\left[\hat{C}_{i,k+1}\right] = \hat{C}_{ik}^2\left(\mathrm{Var}[F_{ik}] + \mathrm{Var}[\hat{f}_k]\right) + \mathrm{Var}\left[\hat{C}_{ik}\right]\hat{f}_k^2$$

with individual development factors $F_{ik} = C_{i,k+1} / C_{ik}$

Results of Mack (3)

$$\mathrm{Var}\left[\hat{C}_{i,k+1}\right] = \hat{C}_{ik}^2\left(\mathrm{Var}[F_{ik}] + \mathrm{Var}[\hat{f}_k]\right) + \mathrm{Var}\left[\hat{C}_{ik}\right]\hat{f}_k^2$$

process error
new observation

estimation error

error carried forward

⇒ easy to program (e.g. in Excel)
⇒ numerically stable

Results of Mack – Numerical Example (1)

Data from case 1:

	1	2	3	4	5	6	7	8	9	10
1	5.012	8.269	10.907	11.805	13.539	16.181	18.009	18.608	18.662	18.834
2	106	4.285	5.396	10.666	13.782	15.599	15.496	16.169	16.704	
3	3.410	8.992	13.873	16.141	18.735	22.214	22.863	23.466		
4	5.655	11.555	15.766	21.266	23.425	26.083	27.067			
5	1.092	9.565	15.836	22.169	25.955	26.180				
6	1.513	6.445	11.702	12.935	15.852					
7	557	4.020	10.946	12.314						
8	1.351	6.947	13.112							
9	3.133	5.395								
10	2.063									

Results of Mack – Numerical Example (2)

Step 1 Individual development factors F_{ik} and CL-factors f_k

	1	2	3	4	5	6	7	8	9	10
1		1,650	1,319	1,082	1,147	1,195	1,113	1,033	1,003	1,009
2		40,425	1,259	1,977	1,292	1,132	0,993	1,043	1,033	
3		2,637	1,543	1,163	1,161	1,186	1,029	1,026		
4		2,043	1,364	1,349	1,102	1,113	1,038			
5		8,759	1,656	1,400	1,171	1,009				
6		4,260	1,816	1,105	1,226					
7		7,217	2,723	1,125						
8		5,142	1,887							
9		1,722								
10										
	f_k	2,999	1,624	1,271	1,172	1,113	1,042	1,033	1,017	1,009

Results of Mack – Numerical Example (3)

Step 2 Calculate variances $\hat{\sigma}_k^2$, $\mathrm{Var}[\hat{f}_k]$ and $\mathrm{Var}[F_{ik}]$

$$\hat{\sigma}_k^2 = \frac{1}{n-k-1}\sum_i C_{ik}(F_{ik} - \hat{f}_k)^2$$

k	1	2	3	4	5	6	7	8	9
σ_k	167	33	26	7,8	11	6,4	1,2	2,8	1,2

$$\mathrm{Var}[\hat{f}_k] = \sigma_k^2 / \sum_j C_{jk}$$

k	1	2	3	4	5	6	7	8	9
$\mathrm{SD}[\hat{f}_k]$	1,13	0,136	0,090	0,025	0,035	0,023	0,005	0,015	0,008

Results of Mack – Numerical Example (4)

$\mathrm{Var}[F_{ik}] = \hat{\sigma}_k^2 \;/\; C_{ik}$

	1	2	3	4	5	6	7	8	9	10
1		5,56	0,13	0,06	0,01	0,01	0,00	0,00	0,00	0,00
2		263,05	0,26	0,13	0,01	0,01	0,00	0,00	0,00	0,00
3		8,18	0,12	0,05	0,00	0,01	0,00	0,00	0,00	0,00
4		4,93	0,10	0,04	0,00	0,01	0,00	0,00	0,00	0,00
5		25,53	0,12	0,04	0,00	0,00	0,00	0,00	0,00	0,00
6		18,43	0,17	0,06	0,00	0,01	0,00	0,00	0,00	0,00
7		50,06	0,28	0,06	0,00	0,01	0,00	0,00	0,00	0,00
8		20,64	0,16	0,05	0,00	0,01	0,00	0,00	0,00	0,00
9		8,90	0,21	0,08	0,01	0,01	0,00	0,00	0,00	0,00
10		13,52	0,18	0,07	0,00	0,01	0,00	0,00	0,00	0,00

Results of Mack – Numerical Example (5)

Step 3 Apply recursion-formula

	1	2	3	4	5	6	7	8	9	10
1										0
2									0	206
3								0	557	623
4							0	232	673	747
5						0	1.191	1.252	1.423	1.469
6					0	1.486	1.810	1.879	1.970	2.002
7				0	923	1.743	2.022	2.096	2.179	2.209
8			0	3.236	3.947	4.703	5.014	5.185	5.302	5.358
9		0	2.553	4.149	4.939	5.657	5.954	6.154	6.272	6.333
10	0	7.935	13.173	16.971	19.907	22.211	23.160	23.931	24.341	24.566

Results of Mack – Numerical Example (6)

A similar procedure is available for the total prediction error:

$$\text{MSEP} \approx \sum_i \left\{ MSEP_i + \hat{C}_{i,ult}^2 \left(\sum_{k \geq i+1} \hat{C}_{k,ult} \right) \sum_{j \geq n+1-i} \frac{\hat{\sigma}_j^2 / \hat{f}_j^2}{\sum_q C_{qj}} \right\}$$

$\Rightarrow$ Outcome: total reserve 52.135

total prediction error 26.909

Results of Mack – Numerical Example (7)

Comparison between Mack and Overdispersed Poisson:

			Mack		*Overdispersed Poisson*	
	Ultimate loss	**Reserve**	**Prediction error**	**Prediction error %**	**Prediction error**	**Prediction error %**
1	18.834	0	0	-	0	-
2	16.858	154	206	134%	556	361%
3	24.083	617	623	101%	1.120	181%
4	24.884	1.636	747	46%	1.775	108%
5	28.927	2.747	1.469	53%	2.231	81%
6	19.501	3.649	2.002	55%	2.440	67%
7	17.749	5.435	2.209	41%	3.124	57%
8	24.019	10.907	5.358	49%	5.032	46%
9	16.045	10.650	6.333	59%	6.075	57%
10	18.402	16.339	24.566	150%	12.987	79%
Total		**52.135**	**26.909**	**52%**	**18.193**	**35%**

Bootstrapping

Second topic: bootstrapping

Predictive distribution - bootstrapping

So: expected loss and prediction error can be estimated.

How about the entire distribution of the estimated reserve?

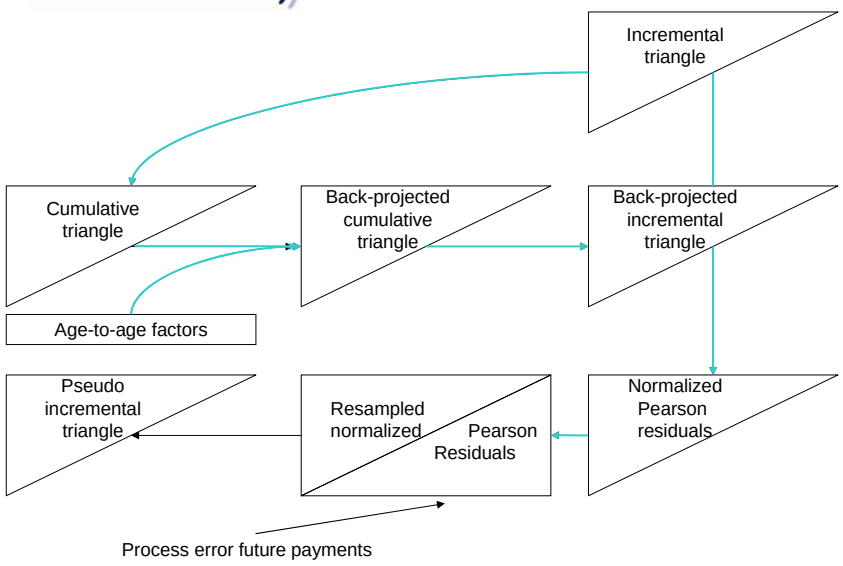
⇒ use bootstrapping-technique (Efron)

- both powerful and simple (spreadsheet)
- sampling with replacement from the observed data
- consistency with the underlying distribution

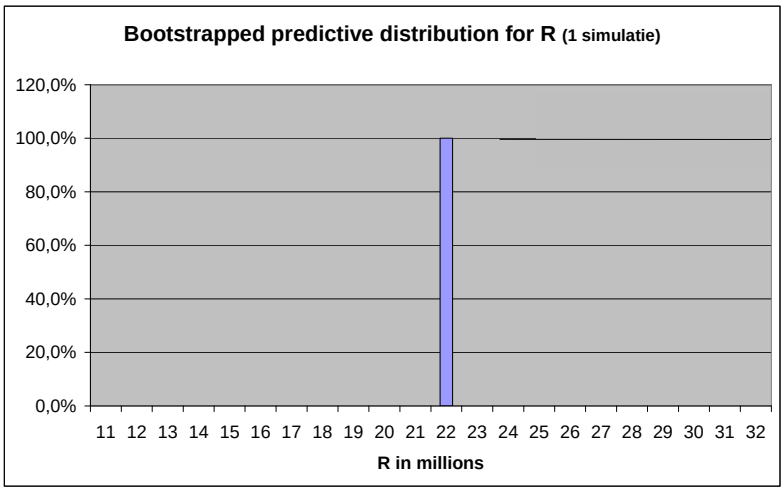
Bootstrapping – Recipe England & Verrall

- calculate development factors (cumulative data)
- obtain fitted values past triangle (cumulative and incremental)
- calculate (adjusted) Pearson residuals,
- start iteration (say 1000 times):
  - resample residuals ⇒ new past triangle of residuals
  - for each cell, calculate as-if observation with residual
  - create cumulative data
  - fit Chain Ladder model
  - calculate projected cumulative payments
  - calculate incremental payments
  - for each cell, simulate payment from process distribution with mean equal to incremental payment of previous step
  - sum simulated payments in future triangle

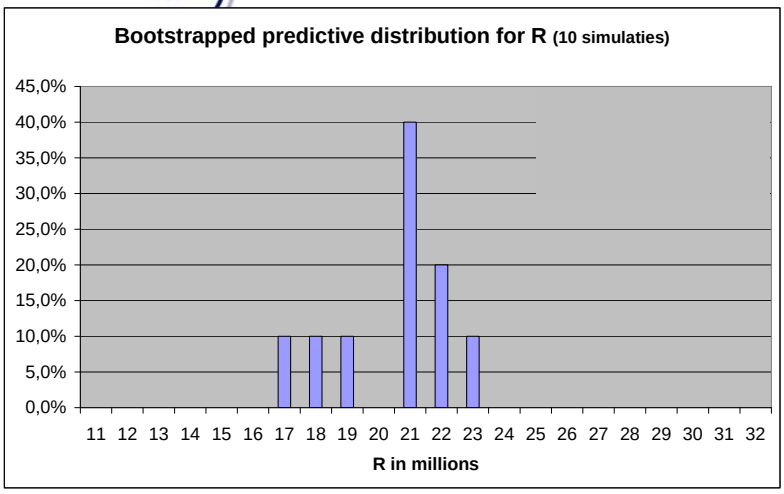
Bootstrapping – Graphical scheme



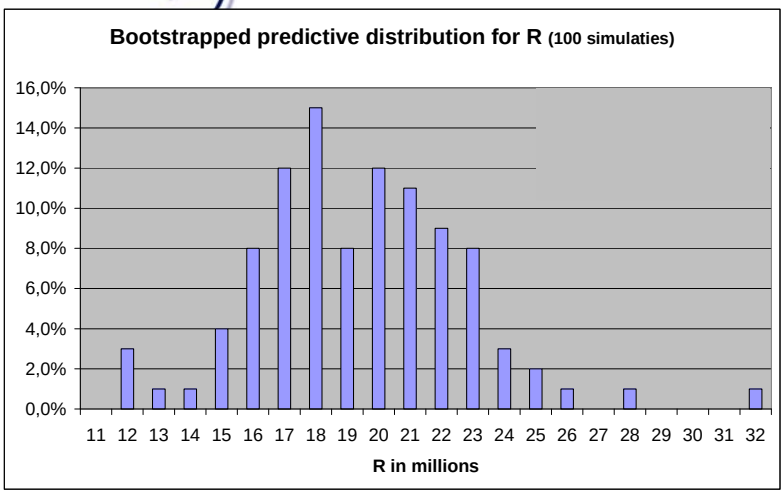
Simulation of future loss – 1 simulation



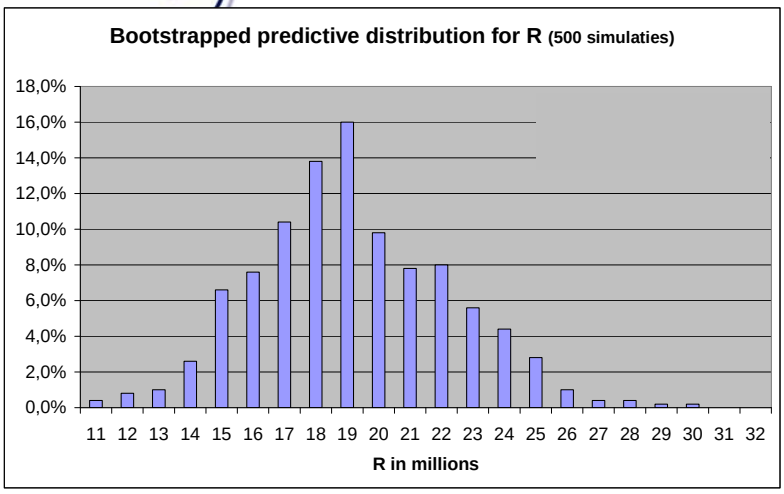
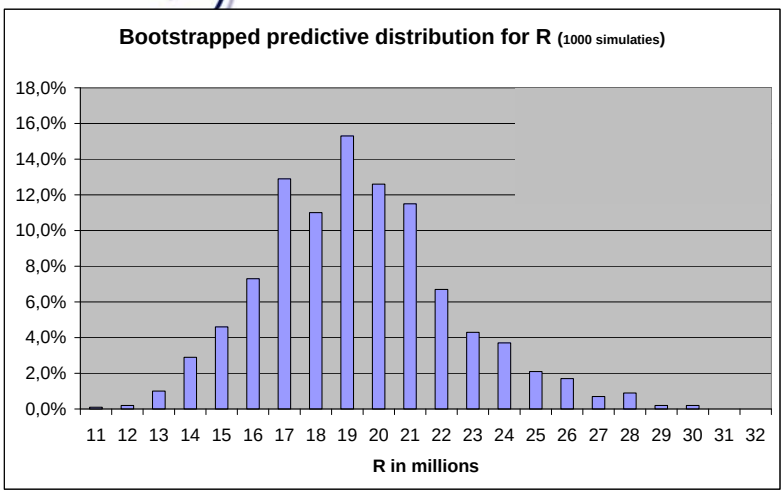
Simulation of future loss – 10 simulations



Simulation of future loss – 100 simulations



Simulation of future loss – 500 simulations



Simulation of future loss – 1000 simulations